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EDDY CURRENTS IN A CONDUCTING PLATE IN PERIODICAL NON-HARMONIC STEADY-STATE

BY

HUGO ROSMAN[*]

“Gheorghe Asachi” Technical University of Iaşi,
Faculty of Electrical Engineering, Energetics
and Applied Informatics

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Abstract. The eddy currents in a conducting plate having a rectangular section, situated in a homogeneous magnetic field, which is a periodical, non-harmonic function of time, are studied. The utilized method, which permits to study the eddy currents in the conducting plate, is based on a symbolic proceeding representing the periodical, non-harmonic signals, through hypercomplex “images”. The hypercomplex moduli of the electromagnetic field’s state vectors are determined, individually, namely those produced by the low, medium and high order harmonics of the external magnetic field. The resultant expressions of the hypercomplex electromagnetic field’s state vectors are obtained applying the superposition theorem having in view that the behaviour of the conducting plate in electromagnetic field is considered a linear one.

Key words: eddy currents; conducting plate with a rectangular section; external periodical non-harmonic and homogeneous magnetic field; hypercomplex symbolic method.

[*] *e-mail*: adi_rotaru2005@yahoo.com

1. Introduction

It is well known that if a massive conductor is placed in a variable in time magnetic field in the conductor are generated currents named *eddy currents* (Foucault). The most frequent cases studied in the specialty literature are those concerning a conducting plate having a rectangular section or a straight cylindrical conductor having a circular section, situated in a homogeneous magnetic field harmonically variable in time (for instance (Rosman *et al*., 1999)).

In what follows the eddy currents generated in a conducting plate, having a rectangular section and situated in a homogeneous magnetic field, are studied, when the magnetic field is variable in time periodically but not harmonically. The conducting plate is considered infinite long, having the width *d* (Fig. 1), the external magnetic field allowing the development in a Fourier series

$$\mathbf{H}_0(t) = \sum_{k=0}^{\infty} \mathbf{H}_0^{'} \cos k\omega t + \sum_{k=0}^{\infty} \mathbf{H}_0^{''} \sin k\omega t. \qquad (1)$$

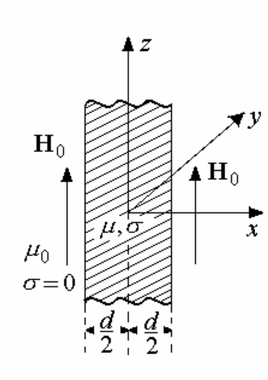

Fig. 1

It is considered that this magnetic field is tangentially oriented to plate's faces.

When the external magnetic field has a harmonically variation in time, the complex r.m.s. values of the electromagnetic field induced in the plate vector are (Mocanu, 1984)

$$\underline{\mathbf{E}}_{\text{int}}(x) = -\mathbf{j}\frac{\underline{\gamma}\underline{H}_0}{\sigma} \cdot \frac{\text{sh}\underline{\gamma}x}{\text{ch}\frac{\underline{\gamma}d}{2}}, \quad \underline{\mathbf{H}}_{\text{int}}(x) = \mathbf{k}\underline{H}_0 \frac{\text{ch}\underline{\gamma}x}{\text{sh}\frac{\underline{\gamma}d}{2}},$$
$$\underline{\mathbf{J}}(x) = -\mathbf{j}\underline{\gamma}\underline{H}_0 \frac{\text{sh}\underline{\gamma}x}{\text{ch}\frac{\underline{\gamma}d}{2}}, \qquad (2)$$

where $\underline{H}_0$ is the complex r.m.s. value of vector

$$\mathbf{H}_0 = \mathbf{k}\sqrt{2}H_0 \sin \omega t \qquad (3)$$

and

$$\underline{\gamma} = \alpha + \text{j}\beta, \quad \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}. \qquad (4)$$

Here $\underline{\gamma}$ represents the complex propagation constant of the electromagnetic wave in the conducting medium, α – the attenuation constant and β – the phase constant ($\alpha = \beta$), μ, σ – the material constants of the conducting medium, considered homogeneous, isotropic and without hereditary phenomenons.

Vectors $\mathbf{E}_{\text{int}}(x, t)$, $\mathbf{H}_{\text{int}}(x, t)$ being known, it is possible to determine Poynting vector, for instance in a point situated on the conducting plate's surface. His complex expression is

$$\underline{S}\left(\pm\frac{d}{2}\right) = \mp\underline{\gamma}\frac{H_0^2}{\sigma}\cdot\frac{\text{sh}\frac{\underline{\gamma}d}{2}}{\text{ch}\frac{\underline{\gamma}d}{2}}. \tag{5}$$

At *low frequencies* it is possible to retain from hyperbolic trigonometric functions series developments

$$\text{sh}\, m = m + \frac{m^3}{6} + \frac{m^5}{120} + \ldots,\ \text{ch}\, m = 1 + \frac{m^2}{2} + \frac{m^4}{24} + \ldots,\ \left(m = \hat{\gamma}x, \hat{\gamma}d/2\right), \tag{6}$$

only the first two terms. In this case relations (2) and (5) become

$$\underline{E}_{\text{int}}(x) = -\frac{\underline{\gamma}\underline{H}_0 x}{3\sigma}\cdot\frac{6+\underline{\gamma}^2x^2}{8+\underline{\gamma}^2d^2},\quad \underline{H}_{\text{int}}(x) = 4\underline{H}_0\frac{2+\underline{\gamma}^2x^2}{8+\underline{\gamma}^2d^2},$$
$$\underline{J}_{\text{int}}(x) = -\frac{\underline{\gamma}\underline{H}_0 x}{3}\cdot\frac{6+\underline{\gamma}^2x^2}{8+\underline{\gamma}^2d^2}, \tag{7}$$

respectively

$$\underline{S}\left(\pm\frac{d}{2}\right) = \mp\frac{\underline{\gamma}^2H_0^2d}{6\sigma}\cdot\frac{24+\underline{\gamma}^2d^2}{8+\underline{\gamma}^2d^2}. \tag{8}$$

At *high frequencies* it is possible to approximate the hyperbolic trigonometric functions with their asymptotic values

$$\text{ch}\, m \approx \text{e}^m,\quad \text{sh}\, m = \text{e}^m, \tag{9}$$

so that relations (2) and (5) become

$$\underline{E}_{\text{int}}(x) = -\frac{\underline{\gamma}\underline{H}_0}{\sigma}\text{e}^{\underline{\gamma}(x-d/2)},\ \underline{H}_{\text{int}}(x) = \underline{H}_0\text{e}^{\underline{\gamma}(x-d/2)},\ \underline{J}(x) = -\underline{\gamma}\underline{H}_0\text{e}^{\underline{\gamma}(x-d/2)}, \tag{10}$$

respectively

$$\underline{S}\left(\pm\frac{d}{2}\right)=\mp\frac{\gamma H_0^2}{\sigma}. \tag{11}$$

In what follows the dispersion *vs*. time of material constants, σ, μ , is neglected.

2. Study Method

It is advantageously to study the eddy currents in periodical non-harmonic steady-state using a symbolic method based on representation of periodical, non-harmonic signals, through hypercomplex images, proposed by B.A. R o z e n f e l d (1949). This method consists in attaching to a periodical non-harmonic signal, which admits the development in Fourier series

$$a(t)=\sum_{k=0}^{\infty}A_k^{'}\cos k\omega t+\sum_{k=0}^{\infty}A_k^{''}\sin k\omega t, \tag{12}$$

a hypercomplex “image”

$$\hat{A}=\sum_{k=0}^{\infty}1_k\,A_k^{'}+\sum_{k=0}^{\infty}\mathrm{j}_k\,A_k^{''}, \tag{13}$$

where functions 1_k , j_k are orthonormalized. The so defined algebra is commutative, representing a direct sum of real number field (generated by 1_0) and the numberable set of complex numbers set (generated by the pair 1_k , j_k), the unity element of this algebra being

$$\sum_{k=0}^{\infty}1_k=1_0. \tag{14}$$

The relations

$$\begin{gathered}1_k^2=1_k,\ \mathrm{j}_k^2=-1_k,\ 1_k\,\mathrm{j}_k=\mathrm{j}_k 1_k=\mathrm{j}_k,\ 1_p 1_q=1_q 1_p=\mathrm{j}_p\mathrm{j}_q=\mathrm{j}_q\mathrm{j}_p=\\ =1_p\,\mathrm{j}_q=\mathrm{j}_p 1_q=0,\ (p\neq q),\end{gathered} \tag{15}$$

are satisfied too.

Having in view that the symbolic identity

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} = \sum_{k=0}^{\infty} \left(\mathrm{j}_k k\omega \right)^m, \quad m \in \mathbb{N}, \tag{16}$$

is verified by the hypercomplex “images”, it results that this symbolic method posses the advantage of “algebrization” of differential operations with respect to time, analogous with the one presented by the symbolic method of complex “images” used in the case of harmonic steady-state study.

A variant of this symbolic method was exposed by the author in a previous paper (Rosman, 2010).

3. Hypercomplex Vectors $\hat{\mathbf{E}}_{\mathrm{int}}(x)$, $\hat{\mathbf{H}}_{\mathrm{int}}(x)$, $\hat{\mathbf{S}}(d/2)$

If the state hypercomplex vectors of an electromagnetic field, in periodical, non-harmonic steady-state, in a point situated at distance $x \in [0, d/2]$ from the conducting plate’s (having a rectangular section) symmetry axis, are denoted with $\hat{\mathbf{E}}_{\mathrm{int}}(x)$ and $\hat{\mathbf{H}}_{\mathrm{int}}(x)$, when the plate is situated in a homogeneous, periodical but non-harmonic magnetic field, having the expression (1), d being the plate’s width, the expressions of these hypercomplex vectors are, from a formal point of view, analogous to relations (2), being sufficiently to substitute in these relations $\underline{\gamma}$ and $\underline{H}_0$ with

$$\hat{\gamma} = \sum_{k=0}^{\infty} \left(1_k \alpha_k + \mathrm{j}_k \beta_k \right) = \sqrt{\sum_{k=0}^{\infty} \mathrm{j}_k k\omega\sigma\mu} \tag{17}$$

and, respectively,

$$\hat{H}_0 = \sum_{k=0}^{\infty} \left(1_k H'_{0k} + \mathrm{j}_k H''_0 \right), \tag{18}$$

where $\hat{\gamma}$ is the hypercomplex propagation constant (Rosman, 1979), $\hat{H}_0$ – the external magnetic field hypercomplex r.m.s. value and

$$\alpha_k = \beta_k = \sqrt{\frac{k\omega\sigma\mu}{2}} \tag{19}$$

– the attenuation (phase) constant of the electromagnetic field wave’s k-th order harmonic. Performing the above substitutions the hypercomplex moduli of the electromagnetic field’s state vectors expressions

$$\hat{E}_{\text{int}}(x) = -\frac{\hat{\gamma}\hat{H}_0}{\sigma}\cdot\frac{\text{sh}\,\hat{\gamma}x}{\text{ch}\,\frac{\hat{\gamma}d}{2}},\quad \hat{H}_{\text{int}}(x) = \hat{H}_0\,\frac{\text{ch}\,\hat{\gamma}x}{\text{ch}\,\frac{\hat{\gamma}d}{2}},\quad \hat{J}(x) = -\hat{\gamma}\hat{H}_0\,\frac{\text{sh}\,\hat{\gamma}x}{\text{ch}\,\frac{\hat{\gamma}d}{2}},\qquad (20)$$

are obtained.

The hypercomplex Poynting vector, in a point situated on the plate surface, is

$$\hat{\mathbf{S}}\left(\frac{d}{2}\right) = \hat{\mathbf{E}}_{\text{int}}\left(\frac{d}{2}\right)\times\hat{\mathbf{H}}^{*}_{\text{int}}\left(\frac{d}{2}\right),\qquad (21)$$

having the hypercomplex modulus

$$\hat{S}\left(\frac{d}{2}\right) = \hat{E}\left(\frac{d}{2}\right)\hat{H}^{*}\left(\frac{d}{2}\right).\qquad (22)$$

In relations (21) and (22) the asterisk characterize the hypercomplex conjugate of an element so that if $\hat{M} = \sum_{k=0}^{\infty} 1_k M_k^{'} + \sum_{k=0}^{\infty} \text{j}_k M_k^{"}$ then $\hat{M}^{*} = \sum_{k=0}^{\infty} 1_k M_k^{'} - \sum_{k=0}^{\infty} \text{j}_k M_k^{"}$ and consequently $\hat{M}\hat{M}^{*} = \sum_{k=0}^{\infty} 1_k \left(M_k^{'2} + M_k^{"2}\right) = M^2$.

As it was proved in a previous paper (Rosman, 1960) it is possible to define, in periodical, non-harmonic steady-state, a *hypercomplex apparent power*

$$\hat{s} = \sum_{k=0}^{\infty} 1_k P_k + \sum_{k=0}^{\infty} \text{j}_k Q_k\ ,\qquad (23)$$

where

$$P = \sum_{k=0}^{\infty} P_k\ ,\quad Q = \sum_{k=0}^{\infty} Q_k\qquad (24)$$

represent the active, respectively the reactive power which correspond to all harmonics. If the hypercomplex apparent power on the conducting plate’s face surface area unit is considered then

$$\hat{s} = \hat{S}\left(\frac{d}{2}\right),\qquad (25)$$

allowing to perform the active power, *P*, and the reactive power, *Q*.

When the external magnetic field's fundamental term frequency is not to high, in his frequencies spectrum may be identified three domains in which the generated eddy currents are, from qualitative point of view, different namely: a) the domain [0, *pf*], with sufficiently low frequencies so that from series developments (6) may be retained only the first two terms; b) the domain [*pf*, *qf*], having only medium frequencies being necessary to take into account the series (6) in their totality; c) the domain [*qf*, ∞) which corresponds to the high order harmonics; the trigonometric functions may be approximated with the asymptotic values (9).

3.1. Case of External Magnetic Field Harmonics Having Low Frequencies

In this case the contribution to the eddy currents of external magnetic field's harmonics of low order may be determined retaining only the two first terms of series (6). Consequently, having in view (7) it results

$$\hat{E}_{\text{int}\,p}(x) = -\frac{4\hat{\gamma}_p^2 x\hat{H}_{0p}}{3\sigma}\cdot\frac{6+\hat{\gamma}_p^2x^2}{8+\hat{\gamma}_p^2d^2},\quad \hat{H}_{\text{int}\,p}(x) = 4\hat{H}_{0p}\frac{2+\hat{\gamma}_p^2x^2}{8+\hat{\gamma}_p^2d^2},$$
$$\hat{J}_p(x) = -\frac{4\hat{\gamma}_p^2 x\hat{H}_{0p}}{3}\cdot\frac{6+\hat{\gamma}_p^2x^2}{8+\hat{\gamma}_p^2d^2}, \tag{26}$$

where

$$\hat{\gamma}_p = \sum_{k=0}^{p}\left(1_k\alpha_k + \mathrm{j}_k\beta_k\right) = \sqrt{\sum_{k=0}^{p}\mathrm{j}_k k\omega\sigma\mu} \tag{27}$$

and

$$\hat{H}_{0p} = \sum_{k=0}^{p}\left(1_k H'_{0p} + \mathrm{j}_k H''_{0p}\right). \tag{28}$$

As well the Poynting vector hypercomplex modulus (s. (8)) is

$$\hat{S}_p\left(\pm\frac{d}{2}\right) = \mp\frac{\hat{\gamma}_p^2 dH_{0p}^2}{6\sigma}\cdot\frac{24+\hat{\gamma}_p^2d^2}{8+\hat{\gamma}_p^2d^2}. \tag{29}$$

Substituting in relations (26) and (29), expression (27) of $\hat{\gamma}_p$ and taking into account relations (14),…,(16), expressions (26) and (29) become

$$\begin{cases} \hat{E}_{\text{int}\,p}(x) = 4x\hat{H}_{0p}\dfrac{\sum_{k=0}^{p} 1_k k^2\omega^2\sigma\mu^2 x^2 + \sum_{k=0}^{p} \mathrm{j}_k 2k\omega\mu}{\sum_{k=0}^{p} 1_k 8 + \sum_{k=0}^{p} \mathrm{j}_k k\omega\sigma\mu d^2}, \\ \hat{H}_{\text{int}\,p}(x) = \hat{H}_{0p}\dfrac{\sum_{k=0}^{p} 1_k 8 + \sum_{k=0}^{p} \mathrm{j}_k 4k\omega\sigma\mu x^2}{\sum_{k=0}^{p} 1_k 8 + \sum_{k=0}^{p} \mathrm{j}_k k\omega\sigma\mu d^2}, \\ \hat{J}_{\text{int}\,p}(x) = 4x\hat{H}_{0p}\dfrac{\sum_{k=0}^{p} 1_k k^2\omega^2\sigma^2\mu^2 x^2 + \sum_{k=0}^{p} \mathrm{j}_k 2k\omega\sigma\mu}{\sum_{k=0}^{p} 1_k 8 + \sum_{k=0}^{p} \mathrm{j}_k k\omega\sigma\mu d^2}, \end{cases} \tag{30}$$

respectively

$$\hat{S}_p\left(\pm\frac{d}{2}\right) = \mp\frac{H_{0p}^2}{6}\cdot\frac{\sum_{k=0}^{p} 1_k k^2\omega^2\sigma^2\mu^2 d^2 - \sum_{k=0}^{p} \mathrm{j}_k 24k\omega\mu d}{\sum_{k=0}^{p} 1_k 8 + \sum_{k=0}^{p} \mathrm{j}_k k\omega\sigma\mu d^2}. \tag{31}$$

Amplifying the fractions from (30) and (31) with the denominator's hypercomplex conjugate (the same), $\sum_{k=0}^{p} 1_k 8 - \sum_{k=0}^{p} \mathrm{j}_k k\omega\sigma\mu d^2$ it results

$$\begin{cases}\hat{E}_{\text{int}\,p}(x)=4x\hat{H}_{0p}\dfrac{\sum\limits_{k=0}^{p}1_k 2k^2\omega^2\sigma\mu^2\left(4x^2-d^2\right)-\sum\limits_{k=0}^{p}\mathrm{j}_k k\omega\mu\left(16+k^2\omega^2\sigma^2\mu^2d^2\right)}{64+\sum\limits_{k=0}^{p}k^2\omega^2\sigma^2\mu^2d^4},\\ \hat{H}_{\text{int}\,p}(x)=\hat{H}_{0p}\dfrac{\sum\limits_{k=0}^{p}1_k\left(64+k^2\omega^2\sigma^2\mu^2d^2x^2\right)+\sum\limits_{k=0}^{p}\mathrm{j}_k 8k\omega\sigma\mu\left(4x^2-d^2\right)}{64+\sum\limits_{k=0}^{p}k^2\omega^2\sigma^2\mu^2d^4},\\ \hat{J}_{\text{int}\,p}(x)=4x\hat{H}_{0p}\dfrac{\sum\limits_{k=0}^{p}1_k 2k^2\omega^2\sigma^2\mu^2\left(4x^2-d^2\right)-\sum\limits_{k=0}^{p}\mathrm{j}_k k\omega\sigma\mu\left(16+k^2\omega^2\sigma^2\mu^2d^2\right)}{64+\sum\limits_{k=0}^{p}k^2\omega^2\sigma^2\mu^2d^4},\end{cases} \tag{32}$$

respectively

$$\hat{S}_p\left(\pm\frac{d}{2}\right)=\mp\frac{H_{0p}^2}{6}\cdot\frac{\sum\limits_{k=0}^{p}1_k 16k^2\omega^2\sigma\mu d^2+\sum\limits_{k=0}^{p}\mathrm{j}_k k\omega\mu d\left(192+k^2\omega^2\sigma^2\mu^2d^4\right)}{64+\sum\limits_{k=0}^{p}k^2\omega^2\sigma^2\mu^2d^4}. \tag{33}$$

Having in view relations (22),…,(25) the expressions of active and reactive power on area unit of plate's surface and on the length unit of this one are

$$P_p=\frac{H_{0p}^2}{3}\cdot\frac{8\omega^2\sigma\mu d\sum\limits_{k=0}^{p}k^2}{64+\omega^2\sigma^2\mu^2d^2\sum\limits_{k=0}^{p}k^2}, \tag{34}$$

respectively

$$Q_p=\frac{H_{0p}^2}{3}\cdot\frac{\omega\mu d\left(192\sum\limits_{k=0}^{p}k+\omega^2\sigma^2\mu^2d^4\sum\limits_{k=0}^{p}k^3\right)}{64+\omega^2\sigma^2\mu^2d^2\sum\limits_{k=0}^{p}k^2}. \tag{35}$$

But (Ryžik et al., 1951)

$$\sum_{k=0}^{p} k = \frac{p(p+1)}{2}, \quad \sum_{k=0}^{p} k^2 = \frac{p(p+1)(2p+1)}{6}, \quad \sum_{k=0}^{p} k^3 = \frac{p^2(p+1)^2}{4}, \quad (36)$$

so that relations (34), (35) become

$$P_p = \frac{H_{0p}^2}{3} \cdot \frac{8p(p+1)(2p+1)\omega^2 \sigma\mu d^2}{384 + p(p+1)(2p+1)\omega^2\sigma^2\mu^2 d^4}, \quad (37)$$

respectively

$$Q_p = \frac{H_{0p}^2}{4} \cdot \frac{p(p+1)\omega\mu d\left[768 + p(p+1)\omega^2\sigma^2\mu^2 d^4\right]}{384 + p(p+1)(2p+1)\omega^2\sigma^2\mu^2 d^4}. \quad (38)$$

3.2. Case of External Magnetic Field Harmonics Having Medium Frequencies

In this case the hypercomplex moduli of electromagnetic field's state vectors are determined using eqs. (20), substituting $\hat{H}_0$ with

$$\hat{H}_{0pq} = \sum_{k=p}^{q} 1_k H_k^{'} + \sum_{k=p}^{q} \mathrm{j}_k H_k^{''} \quad (39)$$

and $\hat{\gamma}$ with

$$\hat{\gamma}_{pq} = \sqrt{\sum_{k=p}^{q} \mathrm{j}_k k\omega\sigma\mu} = \sum_{k=p}^{q} \left(1_k + \mathrm{j}_k\right)\alpha_k \quad (40)$$

so that

$$\hat{E}_{\mathrm{int}\,pq}(x) = \frac{\hat{\gamma}_{pq}\hat{H}_{0pq}}{\sigma} \cdot \frac{\mathrm{sh}\hat{\gamma}_{pq}x}{\mathrm{ch}\frac{\hat{\gamma}_{pq}d}{2}}, \quad \hat{H}_{\mathrm{int}\,pq}(x) = \hat{H}_{0pq}\frac{\mathrm{ch}\hat{\gamma}_{pq}x}{\mathrm{ch}\frac{\hat{\gamma}_{pq}d}{2}},$$
$$\hat{J}_{pq}(x) = \hat{\gamma}_{pq}\hat{H}_{0pq}\frac{\mathrm{sh}\hat{\gamma}_{pq}x}{\mathrm{ch}\frac{\hat{\gamma}_{pq}d}{2}}. \quad (41)$$

In the same time

$$\hat{S}_{pq}\left(\frac{d}{2}\right) = -\frac{\hat{\gamma}_{pq}\hat{H}_{0pq}^{2}}{\sigma}\cdot\frac{\operatorname{sh}\dfrac{\hat{\gamma}_{pq}x}{2}}{\operatorname{ch}\dfrac{\hat{\gamma}_{pq}d}{2}}. \tag{42}$$

In view to obtain the expressions of active power, P_{pq} , and reactive power, Q_{pq} , corresponding to the plate's surface unit and to his length's unit it is necessary to express the fraction $\operatorname{sh}\left(\hat{\gamma}_{\mathrm{pq}}d/2\right)\Big/\operatorname{ch}\left(\hat{\gamma}_{\mathrm{pq}}d/2\right)$ in a canonic hypercomplex form. Having in view (40) it results that

$$\frac{\operatorname{sh}\dfrac{\hat{\gamma}_{pq}d}{2}}{\operatorname{ch}\dfrac{\hat{\gamma}_{pq}d}{2}} = \frac{\operatorname{sh}\left(\sum\limits_{k=p}^{q}1_{\mathrm{k}}\alpha_k d\right) + \operatorname{sh}\left(\sum\limits_{k=p}^{q}\mathrm{j}_{\mathrm{k}}\alpha_k d\right)}{\operatorname{ch}\left(\sum\limits_{k=p}^{q}1_{\mathrm{k}}\alpha_k d\right) + \operatorname{ch}\left(\sum\limits_{k=p}^{q}\mathrm{j}_{\mathrm{k}}\alpha_k d\right)}. \tag{43}$$

Taking into account relations (15), expression (43) becomes

$$\frac{\operatorname{sh}\dfrac{\hat{\gamma}_{pq}x}{2}}{\operatorname{ch}\dfrac{\hat{\gamma}_{pq}d}{2}} = \frac{\sum\limits_{k=p}^{q}1_k\operatorname{sh}\left(\alpha_k d\right)\prod\limits_{\substack{l=p\\ l\neq k}}^{q}\operatorname{ch}\left(\alpha_l d\right) + \sum\limits_{k=p}^{q}\mathrm{j}_k\sin\left(\alpha_k d\right)\prod\limits_{\substack{l=p\\ l\neq k}}^{q}\cos\left(\alpha_l d\right)}{\prod\limits_{l=p}^{q}\operatorname{ch}\left(\alpha_l d\right) + \prod\limits_{l=p}^{q}\cos\left(\alpha_l d\right)}, \tag{44}$$

so relation (42) may be written

$$\hat{S}_{pq}\left(\pm\frac{d}{2}\right)=\mp\sqrt{\frac{\omega\mu}{2\sigma}}H_{0pq}^{2}\times$$

$$\times\frac{\sum_{k=p}^{q}1_k\sqrt{k}\left[\sin(\alpha_k d)\prod_{\substack{l=p\\l\neq k}}^{q}\cos(\alpha_l d)-\operatorname{sh}(\alpha_k d)\prod_{\substack{l=p\\l\neq k}}^{q}\operatorname{ch}(\alpha_l d)\right]}{\prod_{l=p}^{q}\operatorname{ch}(\alpha_l d)+\prod_{l=p}^{q}\cos(\alpha_l d)}- \tag{45}$$

$$-\frac{\sum_{k=p}^{q}\mathrm{j}_k\sqrt{k}\left[\sin(\alpha_k d)\prod_{\substack{l=p\\l\neq k}}^{q}\cos(\alpha_l d)+\operatorname{sh}(\alpha_l d)\prod_{\substack{l=p\\l\neq k}}^{q}\operatorname{ch}(\alpha_l d)\right]}{\prod_{l=p}^{q}\operatorname{ch}(\alpha_l d)+\prod_{l=p}^{q}\cos(\alpha_l d)}.$$

Using relations (23),…,(25), the active power,

$$P_{pq}\left(\frac{d}{2}\right)=\sqrt{\frac{\omega\mu}{2\sigma}}H_{0pq}^{2}\frac{\sum_{k=p}^{q}\sqrt{k}\left[\sin(\alpha_k d)\prod_{\substack{l=p\\l\neq k}}^{q}\cos(\alpha_l d)-\operatorname{sh}(\alpha_k d)\prod_{\substack{l=p\\l\neq k}}^{q}\operatorname{ch}(\alpha_l d)\right]}{\prod_{l=p}^{q}\operatorname{ch}(\alpha_l d)+\prod_{l=p}^{q}\cos(\alpha_l d)}, \tag{46}$$

can be separaterd from the reactive power

$$Q_{pq}\left(\frac{d}{2}\right)=\sqrt{\frac{\omega\mu}{2\sigma}}H_{0pq}^{2}\frac{\sum_{k=p}^{q}\sqrt{k}\left[\sin(\alpha_k d)\prod_{\substack{l=p\\l\neq k}}^{q}\cos(\alpha_l d)+\operatorname{sh}(\alpha_k d)\prod_{\substack{l=p\\l\neq k}}^{q}\operatorname{ch}(\alpha_l d)\right]}{\prod_{l=p}^{q}\operatorname{ch}(\alpha_l d)+\prod_{l=p}^{q}\cos(\alpha_l d)}, \tag{47}$$

both reported on area unit of the conducting plate's lateral surface. Relations (46) and (47) were established taking into account that

$$\begin{cases} \mathrm{ch}\left(1_k \varphi\right) = \mathrm{ch}\ \varphi, \ \ \mathrm{sh}\left(1_k \varphi\right) = 1_k \sin\ \varphi, \\ \mathrm{ch}\left(\mathrm{j}_k \varphi\right) = \cos\ \varphi, \ \ \mathrm{sh}\left(\mathrm{j}_k \varphi\right) = \mathrm{j}_k \sin\ \varphi. \end{cases} \tag{48}$$

3.3. Case of External Magnetic Field Harmonics Having High Frequencies

In such a case the hypercomplex moduli of electromagnetic field state vectors inside the conducting plate may be determined with relations similar to (10) namely

$$\begin{gathered} \hat{E}_{\mathrm{int}\,q}(x) = -\frac{\hat{\gamma}_q \hat{H}_{0q}}{\sigma} \mathrm{e}^{\hat{\gamma}_q (x-d/2)}, \ \ \hat{H}_{\mathrm{int}\,q}(x) = \hat{H}_{0q} \mathrm{e}^{\hat{\gamma}_q (x-d/2)}, \\ \hat{J}_q(x) = -\hat{\gamma}_q \hat{H}_{0q} \mathrm{e}^{\hat{\gamma}_q (x-d/2)}, \end{gathered} \tag{49}$$

where

$$\hat{H}_{0q} = \sum_{k=q}^{\infty} \left(1_k H_0^{'} + \mathrm{j}_k H_0^{''}\right) \tag{50}$$

and

$$\hat{\gamma}_q = \sqrt{\sum_{k=q}^{\infty} \mathrm{j}_k k \omega \sigma \mu} = \sum_{k=q}^{\infty} \left(1_k + \mathrm{j}_k\right) \alpha_k . \tag{51}$$

As regard the Poynting's vector hypercomplex modulus this one is

$$\hat{S}_q \left(\pm \frac{d}{2} \right) = \mp \frac{\hat{\gamma}_q H_{0q}^2}{\sigma} \tag{52}$$

and consequently the expressions of active and reactive powers reported to area unit of conducting plate's lateral surface are

$$P_q = \sqrt{\frac{\omega\mu}{2\sigma}} H_{0q}^2 \sum_{k=q}^{\infty} \sqrt{k}, \ \ Q_k = \sqrt{\frac{\omega\mu}{2\sigma}} H_{0q}^2 \sum_{k=q}^{\infty} \sqrt{k}. \tag{53}$$

Evidently it is possible to retain in the above sums only the contribution of the harmonics $q, q+1, \ldots, q < n < \infty$.

3.4. General Case

In this case are considered all the external magnetic field's harmonics, both the low, medium and high ones. The hypercomplex moduli of electromagnetic field's state vectors as of the Poynting's vector, and the expressions of active and reactive powers reported to area unit of the conducting plate's lateral surface may be obtained applying the superposition theorem because the conducting material used to make up the plate is considered linear. Consequently

$$\begin{cases} \hat{E}_{\text{int}}(x) = \hat{E}_{\text{int}\,p}(x) + \hat{E}_{\text{int}\,pq}(x) + \hat{E}_{\text{int}\,q}(x), \\ \hat{H}_{\text{int}}(x) = \hat{H}_{\text{int}\,p}(x) + \hat{H}_{\text{int}\,pq}(x) + \hat{H}_{\text{int}\,q}(x), \\ \hat{J}(x) = \hat{J}_p(x) + \hat{J}_{pq}(x) + \hat{J}_q(x). \end{cases} \tag{54}$$

As regards the inner electric field's hypercomplex modulus, if in relation (54_1) are substituted $\hat{E}_{\text{int}\,p}(x)$, $\hat{E}_{\text{int}\,pq}(x)$ and $\hat{E}_{\text{int}\,q}(x)$ by expressions (32_1), (41_1) and, respectively, (49_1), it results

$$\hat{E}_{\text{int}}(x) = 4x\hat{H}_{0p}\,\frac{\sum_{k=0}^{p} 1_k 2k^2\omega^2\sigma\mu^2\left(4x^2 - d^2\right) - \sum_{k=0}^{p} \mathrm{j}_k k\omega\mu\left(16 + k^2\omega^2\sigma^2\mu^2 d^2\right)}{64 + \sum_{k=0}^{p} k^2\omega^2\sigma^2\mu^2 d^4} + \\ + \frac{\hat{\gamma}_{pq}\hat{H}_{0pq}}{\sigma}\cdot\frac{\mathrm{sh}\,\hat{\gamma}_{pq}x}{\mathrm{ch}\,\frac{\hat{\gamma}_{pq}d}{2}} - \frac{\hat{\gamma}_q\hat{H}_{0q}}{\sigma}\,\mathrm{e}^{\hat{\gamma}_q(x-d/2)}. \tag{55}$$

In a similar way can be obtained the expression of the inner magnetic field's hypercomplex modulus. It is necessary to substitute in relation (54_2), $\hat{H}_{\text{int}\,p}(x)$, $\hat{H}_{\text{int}\,pq}(x)$, $\hat{H}_{\text{int}\,q}(x)$ by expressions (32_2), (41_2) and, respectively, (49_2) obtaining

$$\hat{H}_{\text{int}}(x) = 4x\hat{H}_{0p}\frac{\sum_{k=0}^{p} 1_k\left(64 + k^2\omega^2\sigma^2\mu^2 d^2 x^2\right) - \sum_{k=0}^{p} \mathrm{j}_k 8k\omega\sigma\mu\left(4x^2 - d^2\right)}{64 + \sum_{k=0}^{p} k^2\omega^2\sigma^2\mu^2 d^4} +$$

$$+\hat{H}_{0pq}\frac{\mathrm{ch}\hat{\gamma}_{pq}x}{\mathrm{ch}\frac{\hat{\gamma}_{pq}d}{2}} + \hat{H}_{0q}\mathrm{e}^{\hat{\gamma}_q(x-d/2)}. \tag{56}$$

Using an analogous proceeding the following expression of eddy currents density hypercomplex modulus may be obtained

$$\hat{J}(x) = 4x\hat{H}_{0p}\frac{\sum_{k=0}^{p} 1_k 2k^2\omega^2\sigma^2\mu^2\left(4x^2 - d^2\right) - \sum_{k=0}^{p} \mathrm{j}_k k\omega\sigma\mu\left(16 + k^2\omega^2\sigma^2\mu^2 d^2\right)}{64 + \sum_{k=0}^{p} k^2\omega^2\sigma^2\mu^2 d^4} -$$

$$-\hat{\gamma}_{pq}\hat{H}_{0pq}\frac{\mathrm{sh}\hat{\gamma}_{pq}x}{\mathrm{ch}\frac{\hat{\gamma}_{pq}d}{2}} - \hat{\gamma}_q\hat{H}_{0q}\mathrm{e}^{\hat{\gamma}_q(x-d/2)}. \tag{57}$$

The expressions of active and reactive power refered to unit area and unit length of conducting plate's lateral surface may be obtained applying this time again the superposition theorem namely

$$P = P_p + P_{pq} + P_q, \quad Q = Q_p + Q_{pq} + Q_q. \tag{58}$$

Having in view relations (34), (46) and (53_1) it results, for the active power, the expression

$$P=\frac{H_{0p}^2}{3}\cdot\frac{8p(p+1)(2p+1)\omega^2\sigma^2\mu^2d^2}{384+p(p+1)(2p+1)\omega^2\sigma^2\mu^2d^4}+$$

$$+\sqrt{\frac{\omega\mu}{2\sigma}}H_{0pq}^2\frac{\sum_{k=p}^{q}\sqrt{k}\left[\sin(\alpha_k d)\prod_{\substack{l=p\\l\neq k}}^{q}\cos(\alpha_l d)-\operatorname{sh}(\alpha_k d)\prod_{\substack{l=p\\l\neq q}}^{q}\operatorname{ch}(\alpha_l d)\right]}{\prod_{l=p}^{q}\operatorname{ch}(\alpha_l d)+\prod_{l=p}^{q}\cos(\alpha_l d)}+\quad(59)$$

$$+\sqrt{\frac{\omega\mu}{2\sigma}}H_{0pq}^2\sum_{k=q}^{n}\sqrt{k}.$$

Similarly, using relations (35), (47) and (53_2) one obtains, for the reactive power, the expression

$$Q=\frac{H_{0p}^2}{4}\cdot\frac{p(p+1)\omega\mu d\left[768+p(p+1)\omega^2\sigma^2\mu^2d^2\right]}{384+p(p+1)(2p+1)\omega^2\sigma^2\mu^2d^2}+$$

$$+\sqrt{\frac{\omega\mu}{2\sigma}}H_{0pq}^2\frac{\sum_{k=p}^{q}\sqrt{k}\left[\sin(\alpha_k d)\prod_{\substack{l=p\\l\neq k}}^{q}\cos(\alpha_l d)+\operatorname{sh}(\alpha_k d)\prod_{\substack{l=p\\l\neq q}}^{q}\operatorname{ch}(\alpha_l d)\right]}{\prod_{l=p}^{q}\operatorname{ch}(\alpha_l d)+\prod_{l=p}^{q}\cos(\alpha_l d)}+\quad(60)$$

$$+\sqrt{\frac{\omega\mu}{2\sigma}}H_{0p}^2\sum_{k=q}^{n}\sqrt{k}.$$

In relations (59) and (60) were considered only the first $n > q$ harmonics of the external magnetic field.

3.5. Particular Case

If the external magnetic field's first harmonic is sufficiently high, the eddy currents induced in the conducting plate, in their totality, generate an electromagnetic field, having as hypercomplex moduli of state vectors expressions (10), where $\hat{\gamma}$ is given by (17) and $\hat{H}$ by (18). Following this proceeding it results

$$\hat{E}_{\text{int}}(x) = -\frac{\hat{\gamma}\hat{H}_0}{\sigma}\text{e}^{\hat{\gamma}(x-d/2)},\ \hat{H}_{\text{int}}(x) = \hat{H}_0\text{e}^{\hat{\gamma}(x-d/2)},\ \hat{J}(x) = -\hat{\gamma}\hat{H}_0\text{e}^{\hat{\gamma}(x-d/2)}. \quad (61)$$

As regards the hypercomplex modulus of Poynting's vector, having in view (11), it results that

$$\hat{S}\left(\pm\frac{d}{s}\right) = \mp\frac{\hat{\gamma}\hat{H}_0^2}{\sigma}. \quad (62)$$

Consequently the active and reactive power refered to conducting plate's lateral surface area unit and length unity are, in absolute values,

$$P = \sqrt{\frac{\omega\mu}{2\sigma}}H_0^2\sum_{k=0}^{n}\sqrt{k},\quad Q = \sqrt{\frac{\omega\mu}{2\sigma}}H_0^2\sum_{k=0}^{n}\sqrt{k}. \quad (63)$$

It is possible to define, in this particular case, a waves *hypercomplex impedance* [8] given by the ratio

$$\hat{\zeta}_0 = \frac{\hat{E}_{\text{int}}(x)}{\hat{H}_{\text{int}}(x)} = -\frac{\hat{\gamma}}{\sigma} = \sqrt{\frac{k\omega\mu}{\sigma}}\text{e}^{-\sum_{k=o}^{\infty}\text{j}_k\pi/4}, \quad (64)$$

which is independent with respect to the depth x. Were taken into account relations (61).

4. Conclusions

The eddy currents generated in a conducting plate having a rectangular section and situated in a homogeneous external magnetic field with periodical, non-harmonic variation in time, oriented tangentially to the plate's faces, are determined.

The utilized study method is based on a symbolic proceeding which permits to attach to each periodical, non-harmonic signal, a hypercomplex "image". The hypercomplex moduli of electromagnetic field's state vectors, $\hat{E}_{\text{int}}(x)$, $\hat{H}_{\text{int}}(x)$, $\hat{J}(x)$, $x \in [0, d/2]$, where d is the conducting plate's width, are determined. In the same time the hypercomplex modulus of Poynting's vector in a point situated on the plate's surface is determined too, which permits to obtain the expressions of the active and reactive powers corresponding to the lateral surface's unit area and length unit of the conducting plate. The determination of these hypercomplex moduli is performed separately for the low, medium and high harmonics of the external magnetic field. The final result is obtained applying the superposition theorem, considering that the plate's material has a linear behaviour in electromagnetic field.

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CURENȚII TURBIONARI ÎNTR-O PLACĂ CONDUCTOARE DE SECȚIUNE DREPTUNGHIULARĂ, ÎN REGIM PERMANENT PERIODIC NEARMONIC

(Rezumat)

Se studiază curenții turbionari induşi într-o placă conductoare de secțiune dreptunghiulară, de un câmp magnetic omogen având o variație în timp periodică dar nearmonică. Metoda utilizată, care permite studiul, în acest caz, al curenților turbionari, se bazează pe un procedeu simbolic bazat pe reprezentarea mărimilor periodice nearmonice prin „imagini" hipercomplexe. Se determină modulii hipercomplecşi ai vectorilor de stare ai câmpului electromagnetic indus în interiorul plăcii, individual, pentru armonicile joase, medii şi, respectiv, înalte ale câmpului magnetic exterior. Expresiile rezultante ale vectorilor de stare hipercomplecşi ai câmpului se obțin utilizând teorema suprapunerii, considerând că mediul conductor al plăcii se comportă liniar în câmpul electromagnetic.