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EIGENVALUE DETERMINATION FOR A RECTANGULAR WAVEGUIDE WITH TWO DIELETRICS IN TE MODE

BY

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Abstract. The solutions obtained for the electric and magnetic field components in a rectangular closed waveguide with metal walls and two longitudinally disposed dielectric layers are determined. The waveguide eigenvalues, complex propagation constant, critical frequency and wavelength are determined too and the practical mode of excitation is established for some numerically analysed cases.

Key words: closed waveguides; wave propagation; separation of variables method.

1. Introduction

High frequency electromagnetic energy, beginning with the microwave range, is transmitted with waveguides, because transmission lines, consisting of two conductor systems, become inefficient at this frequency range due to increased conductor and dielectric losses. Typical applications of waveguides are encountered in TV transmitters, radar installations and satellite communications.

Technical textbooks present comprehensive studies of both one conductor and two conductor rectangular and cylindrical waveguides, as well as

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dielectric or dielectric–conductor waveguides (Miner, 1996; Sadiku, 1995; Petrescu, 2010). The study is almost always limited to the case of uniform waveguides with homogeneous dielectric and simple geometry. Papers published in scientific journals study mostly the case of propagation in waveguides that present a discontinuity of the dielectric properties along the direction of propagation (Rodriguez *et al.*, 2005; Chambers, 1953). In our previous studies closed waveguides, as well as open dielectric–conductor uniform waveguides, were investigated (Petrescu, 2005).

The present paper investigates a rectangular uniform waveguide with two layers of dielectric. The separation of variables method is used in order to solve the Helmholtz differential equation with partial derivatives satisfied by the field components in sinusoidal steady state. The non-linear algebraic equation satisfied by the separation constants is established and solved numerically for several frequencies in the microwave range. The electric and magnetic field components are determined and, based on the field patterns, the practical mode of exciting the rectangular waveguide for a particular frequency is established.

2. Physical Model of the Two Dielectric Waveguide

An one conductor rectangular waveguide with perfectly conducting walls and two layers of ideal dielectric inside is considered (Fig.1). The waveguide axis is oriented in direction Oz-axis. The two dielectric layers, separated by a plane surface parallel with the yOz-plane, have the material constants ε_1, μ_1, and ε_2, μ_2, respectively, and the thicknesses, d_1 and d_2. The waveguide is excited in the TE mode ($H_z \neq 0$, $E_z = 0$), at an angular frequency $\omega = 2\pi f$.

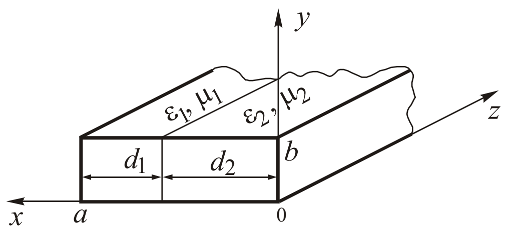

Fig. 1 – Waveguide model.

In sinusoidal steady state the complex r.m.s. value of the magnetic field, $\underline{H}_z$, satisfies the Helmholtz differential equation with partial derivatives

$$\frac{\partial^2 \underline{H}_z}{\partial x^2} + \frac{\partial^2 \underline{H}_z}{\partial y^2} + \underline{k}_c^2 \underline{H}_z = 0\,, \tag{1}$$

where

$$\underline{k}_c^2 = \underline{\gamma}^2 + \omega^2 \mu\varepsilon - \mathrm{j}\omega\mu\sigma \tag{2}$$

and $\underline{\gamma} = \alpha + \mathrm{j}\beta$ is the complex propagation constant in the waveguide. If the two dielectrics are perfect, then $\underline{\gamma} = \mathrm{j}\beta$ and the conductivity is $\sigma = 0$, so that

$$k_c^2 = \omega^2 \mu\varepsilon - \beta^2 . \tag{3}$$

Using the separation of variables method, $\underline{H}_z = X(x)Y(y)$, and, substituting in (1), two ordinary differential equations are obtained namely

$$\frac{X''}{X} = -k_x^2 \, ; \; \frac{Y''}{Y} = -k_y^2 , \tag{4}$$

with $k_x^2 + k_y^2 = k_c^2$, k_x and k_y being separation constants to be further determined from the boundary and interface matching conditions.

The general expression for $\underline{H}_z(x,y,z)$, defined for the domains i = 1, $x \in [d_2, a]$, and i = 2, $x \in [0, d_2]$, is

$$\underline{H}_z(x,y,z) = \left(A_i \sin k_{x_i} x + B_i \cos k_{x_i} x\right)\left(C_i \sin k_{y_i} y + D_i \cos k_{y_i} y\right)\mathrm{e}^{-\underline{\gamma}_i z} . \tag{5}$$

The constants $k_{x_1}, k_{x_2}, k_{y_1}, k_{y_2}, \underline{\gamma}_1, \underline{\gamma}_2$ have different expressions due to the different material properties.

The other components of the magnetic and electric field may be calculated using the relations (Miner, 1996)

$$\underline{H}_x = -\frac{\underline{\gamma}}{k_c^2}\cdot\frac{\partial \underline{H}_z}{\partial x}; \; \underline{H}_y = -\frac{\underline{\gamma}}{k_c^2}\cdot\frac{\partial \underline{H}_z}{\partial y}; \; \underline{E}_x = -\frac{\mathrm{j}\omega\mu}{k_c^2}\cdot\frac{\partial \underline{H}_z}{\partial y}; \; \underline{E}_y = \frac{\mathrm{j}\omega\mu}{k_c^2}\cdot\frac{\partial \underline{H}_z}{\partial x} . \tag{6}$$

In the assumption that the waveguide has perfectly conducting walls, the tangential component of the electric field must be zero near these boundaries.

The condition $\underline{E}_x\big|_{y=0} = 0$ leads to $C_1 = C_2 = 0$ and the relation $\underline{E}_x\big|_{y=b} = 0$ gives the expression for the separation constant $k_{y_1} = k_{y_2} = k_y = n\pi/b$, $n \in \mathbb{N}$. Likewise, $\underline{E}_y\big|_{x=0} = 0$ leads to A_2=0 and $\underline{E}_y\big|_{x=a} = 0$ gives the relation $A_1' \cos k_{x_1} a - B_1' \sin k_{x_1} a = 0$, so that

$$B_1' = A_1' \operatorname{cotan} k_{x_1} a , \tag{7}$$

where $A_1' = A_1 D_1, B_1' = B_1 D_1, C_2' = B_2 D_2$.

The continuity conditions on the surface $x = d_2$ give five more relations namely

a) $\varepsilon_1 \underline{E}_{x_1}\Big|_{x=d_2} = \varepsilon_2 \underline{E}_{x_2}\Big|_{x=d_2}$ leads to

$$\frac{\varepsilon_1\mu_1 k_y}{k_{c_1}^2}(A_1^{'}\sin k_{x_1}d_2 + B_1^{'}\cos k_{x_1}d_2)\sin k_y y = \frac{\varepsilon_2\mu_2 k_y}{k_{c_2}^2}C_2^{'}\cos k_{x_2}d_2 \sin k_y y\ ; \quad (8)$$

b) $\underline{E}_{y_1}\Big|_{x=d_2} = \underline{E}_{y2}\Big|_{x=d_2}$ leads to

$$\frac{\mu_1 k_{x_1}}{k_{c_1}^2}(A_1^{'}\cos k_{x_1}d_2 - B_1^{'}\sin k_{x_1}d_2)\cos k_y y = -\frac{\mu_2 k_{x_2}}{k_{c_2}^2}C_2^{'}\sin k_{x_2}d_2 \cos k_y y\ ; \quad (9)$$

c) $\underline{H}_{z_1}\Big|_{x=d_2} = \underline{H}_{z_2}\Big|_{x=d_2}$ leads to

$$(A_1^{'}\sin k_{x_1}d_2 + B_1^{'}\cos k_{x_1}d_2)\cos k_y y = C_2^{'}\cos k_{x_2}d_2 \cos k_y y\ ; \quad (10)$$

d) $\underline{H}_{y_1}\Big|_{x=d_2} = \underline{H}_{y_2}\Big|_{x=d_2}$ leads to

$$\frac{\underline{\gamma}_1}{k_{c_1}^2}(A_1^{'}\sin k_{x_1}d_2 + B_1^{'}\cos k_{x_1}d_2)k_y \sin k_y y = \frac{\underline{\gamma}_2}{k_{c_2}^2}C_2^{'}\cos k_{x_2}d_2 k_y \sin k_y y\ ; \quad (11)$$

e) $\mu_1 \underline{H}_{x_1}\Big|_{x=d_2} = \mu_2 \underline{H}_{x_2}\Big|_{x=d_2}$ leads to

$$\frac{\mu_1\underline{\gamma}_1 k_{x_1}}{k_{c_1}^2}(A_1^{'}\cos k_{x_1}d_2 - B_1^{'}\sin k_{x_1}d_2)\cos k_y y = -\frac{\mu_2\underline{\gamma}_2 k_{x_2}}{k_{c_2}^2}C_2^{'}\sin k_{x_2}d_2 \cos k_y y\ . \quad (12)$$

Performing the ratio of relations (9) and (12) one obtains

$$\underline{\gamma}_1 = \underline{\gamma}_2, \quad (13)$$

meaning that the propagation constant, $\underline{\gamma}$, is the same in both dielectrics, which was to be expected. Relation (13) may be rewritten, taking into account expression (2), resulting

$$k_{c1}^2 = k_{c2}^2 - \omega^2(\mu_2\varepsilon_2 - \mu_1\varepsilon_1). \quad (14)$$

The ratio of relations (8) and (11) leads to $\mu_1\varepsilon_1 = \mu_2\varepsilon_2$, a condition that can be attained only in particular cases. To overcome the previous restriction, the condition $n = 0$, ($k_y = 0$), may be imposed; in this case relations (8) and (11) are of the form 0=0. This also means that $k_{c_1} = k_{x_1}$, $k_{c_2} = k_{x_2}$.

Thus the pure TE wave in the waveguide with two dielectric layers is characterized by

$$\underline{H}_z(x,y,z) = \begin{cases} \left(A_1^{'} \sin k_{x_1} x + B_1^{'} \cos k_{x_1} x\right) \mathrm{e}^{-\underline{\gamma} z}, \; x \in [d_2, a]; \\ C_2^{'} \cos k_{x_2} x \,\mathrm{e}^{-\underline{\gamma} z}, \; x \in [0, d_2]. \end{cases} \tag{15}$$

Using relations (6) the other components of the electric and magnetic field are $\underline{E}_x = 0$, $\underline{H}_y = 0$,

$$\underline{H}_x(x,y,z) = \begin{cases} -\dfrac{\underline{\gamma}}{k_{x_1}} \left(A_1^{'} \cos k_{x_1} x - B_1^{'} \sin k_{x_1} x\right) \mathrm{e}^{-\underline{\gamma} z}, \; x \in [d_2, a]; \\ \dfrac{\underline{\gamma}}{k_{x_2}} C_2^{'} \sin k_{x_2} x \,\mathrm{e}^{-\underline{\gamma} z}, \; x \in [0, d_2]; \end{cases} \tag{16}$$

$$\underline{E}_y(x,y,z) = \begin{cases} \dfrac{\mathrm{j}\omega\mu_1}{k_{x_1}} \left(A_1^{'} \cos k_{x_1} x - B_1^{'} \sin k_{x_1} x\right) \mathrm{e}^{-\underline{\gamma} z}, \; x \in [d_2, a]; \\ -\dfrac{\mathrm{j}\omega\mu_2}{k_{x_2}} C_2^{'} \sin k_{x_2} x \,\mathrm{e}^{-\underline{\gamma} z}, \; x \in [0, d_2]. \end{cases} \tag{17}$$

Using relations (10) and (7) it results the constant

$$C_2^{'} = \frac{\sin k_{x_1} d_2 + \operatorname{cotan} k_{x_1} a \cos k_{x_1} d_2}{\cos k_{x_2} d_2} A_1^{'} . \tag{18}$$

Substituting (7) and (18) in (12) the relation

$$\frac{\mu_1 k_{x_1}}{\mu_2 k_{x_2}} \cdot \frac{k_{c_2}^2}{k_{c_1}^2} = -\frac{(\sin k_{x_1} d_2 + \operatorname{cotan} k_{x_1} a \cos k_{x_1} d_2) \sin k_{x_2} d_2}{\cos k_{x_2} d_2 (\cos k_{x_1} d_2 - \operatorname{cotan} k_{x_1} a \sin k_{x_1} d_2)} . \tag{19}$$

Is obtained.

Taking into account that $k_{c_1} = k_{x_1}$, $k_{c_2} = k_{x_2}$, equation (14) gives the relationship between the separation constants k_{x_2} and k_{x_1},

$$k_{x_2} = k_{c_2} = \sqrt{k_{c_1}^2 + \omega^2(\mu_2\varepsilon_2 - \mu_1\varepsilon_1)} = \sqrt{k_{x_1}^2 + \omega^2(\mu_2\varepsilon_2 - \mu_1\varepsilon_1)} . \tag{20}$$

Finally, substituting (20) in (19) it results the relation

$$\frac{\mu_1}{\mu_2}\cdot\frac{\sqrt{k_{x_1}^2+\omega^2(\mu_2\varepsilon_2-\mu_1\varepsilon_1)}}{k_{x1}}=-\frac{\mathrm{tg}\sqrt{k_{x_1}^2+\omega^2(\mu_2\varepsilon_2-\mu_1\varepsilon_1)}d_2}{\mathrm{tg}\,k_{x_1}d_1}\,. \tag{21}$$

The previous relation represents a non-linear algebraic equation, the solution of which is the separation constant, k_{x_1}. Using (2) the complex propagation constant the expression

$$\underline{\gamma}=\sqrt{k_{x_1}^2-\omega^2\mu_1\varepsilon_1}=\mathrm{j}\beta\,. \tag{22}$$

is obtained.

Propagation occurs if $\omega>\omega_{\mathrm{cr}}$, where $\omega_{\mathrm{cr}}=k_{x_1}\big/\sqrt{\mu_1\varepsilon_1}=k_{c_1}\big/\sqrt{\mu_1\varepsilon_1}$, this expression being similar to the one obtained in the case of a rectangular waveguide filled with a homogeneous dielectric with the material constants ε_1, μ_1 (Petrescu, 2010).

The wavelength in the two dielectrics is

$$\lambda_g=\frac{2\pi}{\beta}=\frac{2\pi}{\sqrt{\omega^2\mu_1\varepsilon_1-k_{x_1}^2}}\,. \tag{23}$$

3. Results and Discussions

In order to study the frequency behaviour of the waveguide with two vertical dielectric layers, several simulations were made in two cases: $d_2 = a/4$ and $d_2 = a/2$. The numerical values used in the tests were a = 2 cm, b = 1 cm, $\mu_1 = \mu_2 = \mu_0$, ε_{r1} = 1, ε_{r2} = 2.1 (Teflon). The frequencies were considered in the range 10 GHz...50 GHz, correlated with the waveguide dimensions. In each case, for every frequency, the non-linear eq. (21) was solved numerically in MATLAB, in order to determine the value of the separation constant k_{x_1}. Since multiple solutions are obtained in each case, the critical frequency, $f_{\mathrm{cr}}=\omega_{\mathrm{cr}}/2\pi$, was computed for each value of k_{x_1} in order to determine if the wave with the considered frequency, f, satisfies the condition $f>f_{\mathrm{cr}}$. The guide wavelength was calculated and the field components H_z, H_x, E_y were plotted in order to verify that the obtained solution satisfies the boundary and continuity conditions, and also to establish the practical means of excitation. Some of the obtained numerical results are presented in Table 1. The numerical values for k_{x_1} are given in ascending order until the first solution that renders $f<f_{\mathrm{cr}}$ is met.

Table 1
Separation Constant and Excitation Mode in the Case $d_2 = a/2$

f GHz	k_{x_1}	f_{cr} GHz	Propagation	Electric or magnetic dipole antenna position, [cm]	λ_g cm
10	280.35	>10	No	–	–
20	163.82	7.8	Yes	x_0=1.04	1.63
	356.75	17.03	Yes	x_0=1.56	2.86
	558.25	>20	No	–	–
30	273.06	13.03	Yes	x_0=1.42	1.11
	449.57	21.46	Yes	x_0=1.65	1.43
	643.43	>30	No	–	–
40	317.01	15.13	Yes	x_0=1.5	0.81
	559.52	26.71	Yes	$x_0^{(1)}=1.16;\ x_0^{(2)}=1.72$	1.01
	726.28	34.67	Yes	$x_0^{(1)}=1.35;\ x_0^{(2)}=1.78$	1.50
	927.27	>40	No	–	–
50	114.47	5.46	Yes	x_0=1	0.6
	360.04	17.19	Yes	x_0=1.56	0.64
	836.78	39.95	Yes	$x_0^{(1)}=1.06;\ x_0^{(2)}=1.44$	1
	1,005.1	47.99	Yes	$x_0^{(1)}=1.22;\ x_0^{(2)}=1.53;\ x_0^{(3)}=1.84$	2.14
	1,208.1	>50	No	–	–

For comparison the same rectangular waveguide filled with air (homogeneous dielectric) and excited in the TE_{10} mode ($k_y = 0, k_c = k_x = \pi/a = 157.07$) has a critical frequency of $f_{cr}^{(1)} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\cdot\frac{1}{2a} =$ $= 7.5$ GHz (Petrescu, 2010), while if it is filled with Teflon (ε_r = 2.1), the critical frequency is $f_{cr}^{(2)} = 5.17$ GHz. The guide wavelength, λ_g, for the case of the air filled waveguide, having the expression (Petrescu, 2010)

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\omega^2\mu\varepsilon - k_x^2}}, \tag{24}$$

varies between 4.54 cm and 0.61 cm, in the frequency range 10 GHz...50 GHz.

The analysis of the numerical data shows that the pure TE mode can propagate in the waveguide presented in Fig.1 only for the frequencies above 10 GHz. For each frequency in the test range there are up to four TE modes (at most four acceptable solutions for k_{x_1}), but most frequently there are one or two modes. For a given frequency, the wavelength of the TE wave increases for increasing values of k_{x_1}. The critical frequency also increases for increasing values of k_{x_1}, so that, although eq. (21) has an infinite, but numberable number of solutions, only one, two, three, or at most four solutions satisfy the condition $f > f_{\mathrm{cr}}$. As the frequency increases, the number of achievable pure TE modes for a given value of f, also increases, but is limited in both analysed cases, $d_2=a/4$ and $d_2=a/2$.

The practical way to excite the TE mode in the waveguide with two vertical dielectric layers (Fig.1) may be established by inspecting the plots of the field components $H_z(x)$, $H_x(x)$, $E_y(x)$.The fundamental (or dominant) TE mode can be excited by placing an electric dipole antenna oriented in the Oy-axis direction in the position x_0, in the air region, where $E_y(x)$ has a maximum or minimum value, or by placing in the same position a magnetic dipole antenna with its axis parallel to Ox-axis.

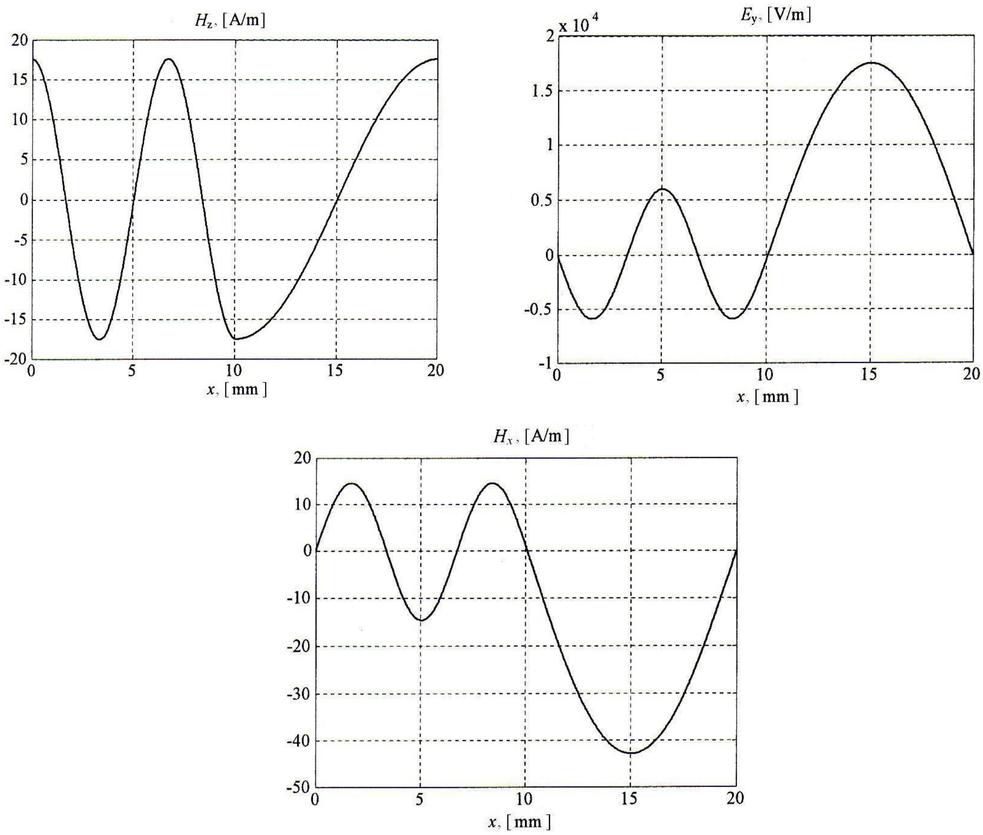

Fig. 2 – Field components corresponding to the case presented in Table 1, line 8.

The plots obtained in the case $d_2 = a/2$, $f = 40$ GHz, $k_{x_1} = 317.01$ are presented in Fig.2. The extreme value corresponding to the air region, x_0, which appears in the plots of $E_y(x)$ and $H_x(x)$, represents the position where the electric or magnetic dipole antenna, having the orientation presented above, may be placed.

4. Conclusions

Rectangular waveguides filled longitudinally with two dielectrics may be completely analysed using the separation of variables method and imposing the boundary and interface conditions. The device acts as a high-pass filter allowing the propagation of waves with frequencies higher than a critical frequency. A limited number of pure TE modes (sometimes only one) can be obtained for each frequency $f > f_{\mathrm{cr}}$. The number of TE modes is given by the number of solutions of the non-linear algebraic equation satisfied by the separation constant k_{x_1}. The waveguide can be excited in the desired TE mode (for a given frequency f and a corresponding guide wavelength) using small electric or magnetic dipolar antennas placed in the positions where the field components E_y or H_x exhibit extreme values.

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DETERMINAREA VALORILOR PROPRII PENTRU UN GHID DE UNDĂ RECTANGULAR CU DOUĂ STRATURI DIELECTRICE FUNCȚIONÂND ÎN MODUL TE

(Rezumat)

Se stabilesc soluțiile analitice, obținute cu ajutorul metodei separării variabilelor, pentru componentele câmpului electromagnetic în interiorul unui ghid de undă metalic rectangular, cu două straturi dielectrice, excitat în modul TE. Se stabileşte şi se rezolvă numeric ecuația algebrică neliniară satisfăcută de valorile proprii, determinându-se frecvența critică, constanta complexă de propagare şi modul practic de excitare al ghidului pentru câteva cazuri numerice analizate.